Fundamentals of Boolean Algebra

- Basic Postulates
  - Postulate 1: Definition. A Boolean algebra is a closed algebraic system and 2 operators $\cdot$ and $+$;
    - for every $a$ and $b$ in set $K$
    - $a \cdot b$ belongs to $K$ and $a+b$ belongs to $K$
  - Postulate 2: Existence of 1 and 0 elements
    - (a) $a+0 = a$
    - (b) $a \cdot 1 = a$

- Postulate 3: Commutativity of the $+$ and $\cdot$ Operations.
  - (a) $a+b = b+a$
  - (b) $a \cdot b = b \cdot a$

- Postulate 4: Associativity of the $+$ and $\cdot$ Operations.
  - (a) $a+(b+c) = (a+b)+c$
  - (b) $a \cdot (b \cdot c) = (a \cdot b) \cdot c$

Agenda

- Fundamentals of Boolean Algebra
- Switching Function
- Switching Circuit
- Analysis of Combinational Circuit
- Synthesis of Combinational Logic Circuit
**Fundamentals of Boolean Algebra**

- **Postulate 5**: Distributivity of $+$ over $\cdot$ and $\cdot$ over $+$
  - (a) $a+(b.c) = (a+b).(a+c)$
  - (b) $a.(b+c) = (a.b)+(a.c)$

- **Postulate 6**: Existence of the complement
  - (a) $a+(\neg a) = 1$
  - (b) $a.(\neg a) = 0$

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**Theorem 1. Idempotency**

- (a) $a+a = a$
- (b) $a.a = a$

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**Theorem 2. Null elements for $+$ and $\cdot$ operators**

- (a) $a+1 = 1$
- (b) $a.0 = 0$

\[
\begin{align*}
  a + 1 &= (a + 1) \quad \text{[P2(b)]} \\
  &= 1 \cdot (a + 1) \quad \text{[P3(b)]} \\
  &= (a + \bar{a})(a + 1) \quad \text{[P6(a)]} \\
  &= a + \bar{a} \cdot 1 \quad \text{[P5(a)]} \\
  &= a + \bar{a} \quad \text{[P2(b)]} \\
  &= 1 \quad \text{[P6(a)]}
\end{align*}
\]

---

**Theorem 3. Involution**

- $\neg(\neg a) = a$

**Theorem 4. Absorption**

- (a) $a+ab = a$
- (b) $a.(a+b) = a$

\[
\begin{align*}
  a + ab &= a \cdot 1 + ab \quad \text{[P2(b)]} \\
  &= a(1 + b) \quad \text{[P5(b)]} \\
  &= a(b + 1) \quad \text{[P3(b)]} \\
  &= a \cdot 1 \quad \text{[T2(a)]} \\
  &= a \quad \text{[P2(b)]}
\end{align*}
\]

\[
\begin{align*}
  a + a &= (X + Y) + (X + Y)Z = X + Y \quad \text{[T4(a)]} \\
  &= a(1 + b) \quad \text{[P5(b)]} \\
  &= a(b + 1) \quad \text{[P3(b)]} \\
  &= a \cdot 1 \quad \text{[T2(a)]} \\
  &= a \quad \text{[P2(b)]}
\end{align*}
\]
Theorem 5.

(a) \(a + a\bar{b} = a + b\).
(b) \(a\bar{a} + b = ab\).

Proof. Part (a) of the theorem is proved as follows:
\[
\begin{align*}
  a + a\bar{b} &= (a + a)\bar{a} + b \\
  &= 1 \cdot (a + b) \\
  &= (a + b) \cdot 1 \\
  &= (a + b) \\
  &= a + b
\end{align*}
\]

\[
B + A\bar{b}\bar{c} = B + AC \quad [T5(a)]
\]

\[
\bar{y}(x + y + z) = \bar{y}(x + z) \quad [T5(b)]
\]

\[
\bar{y}(x + y + z) = \bar{y}(x + z) \quad [T5(a)]
\]

\[
AB + (\bar{A}\bar{B})\bar{C} = AB + C\bar{B} \quad [T5(b)]
\]

Theorem 6.

(a) \(ab + \bar{a}b = a\).
(b) \((a + b)(a + \bar{b}) = a\).

Proof. Part (a) of the theorem is proved as follows:
\[
\begin{align*}
  ab + \bar{a}b &= a(b + \bar{b}) \\
  &= a \cdot 1 \\
  &= a \\
  &= a
\end{align*}
\]

\[
ABC + A\bar{B}C = AC \quad [T6(a)]
\]

Theorem 7.

(a) \(ab + abc = ab + ac\).
(b) \((a + b)(a + \bar{b} + c) = (a + b)(a + c)\).

Proof. Part (a) of the theorem is proved as follows:
\[
\begin{align*}
  ab + abc &= a(b + bc) \\
  &= a(b + c) \\
  &= ab + ac \\
  &= a(b + c)
\end{align*}
\]

\[
\begin{align*}
  xy + x\bar{y}(\bar{w} + \bar{z}) &= x\bar{y} + x\bar{w} + x\bar{z} \\
  &= xy + x\bar{w} + x\bar{z} \\
  &= xy + x\bar{w} + x\bar{z}
\end{align*}
\]

\[
(x\bar{y} + z)(w + x\bar{y} + \bar{z}) = (x\bar{y} + z)(w + x\bar{y}) \quad [T7(b)]
\]
Theorem 8. DeMorgan’s theorem

(a) \( \overline{a + h} = \overline{a} \cdot \overline{h} \).
(b) \( \overline{a \cdot h} = \overline{a} + \overline{h} \).

Complement the expression \( a + bc \).

\[
\begin{align*}
\overline{a + b \cdot c} &= \overline{a} + b \cdot \overline{c} \\
&= \overline{a} + \overline{b} \cdot \overline{c} \\
&= \overline{a} \cdot \overline{b} \\
&= \overline{a} \cdot \overline{b} + \overline{a} \cdot \overline{c}
\end{align*}
\]

Note that: \( \overline{a + b \cdot c} \neq \overline{a} \cdot \overline{b} + \overline{c} \)

Complement the expression \( a(b + z(x + \overline{a})) \), and simplify the result so that the only complemented terms are individual variables.

\[
\begin{align*}
\overline{a(b + z(x + \overline{a}))} &= \overline{a} + (\overline{b + z(x + \overline{a})}) \quad [T8(b)] \\
&= \overline{a} + \overline{b} \cdot \overline{z(x + \overline{a})} \quad [T8(a)] \\
&= \overline{a} + \overline{b} \overline{(z + x) + \overline{a}} \quad [T8(b)] \\
&= \overline{a} + \overline{b} \overline{(z + \overline{x}) + \overline{a}} \quad [T3] \\
&= \overline{a} + \overline{b} \overline{z + \overline{x}} \quad [T5(a)]
\end{align*}
\]

Theorem 9. Consensus

(a) \( ab + \overline{a}c + bc = ab + \overline{a}c \).
(b) \( (a + h)(\overline{a} + c)(\overline{b} + c) = (a + h)(\overline{a} + c) \).

\[
\begin{align*}
ab + \overline{a}c + bc &= ab + \overline{a}c + bc \quad [P2(b)] \\
&= ab + \overline{a}c + (a + \overline{a})bc \quad [P6(a)] \\
&= ab + \overline{a}c + abc + ab \overline{c} \quad [P5(b)] \\
&= (ab + ahc) + (\overline{a}c + \overline{a}eb) \quad [T4(a)] \\
&= ab + \overline{a}c
\end{align*}
\]
The results are valid for any Boolean algebra is often referred to as switching algebra.

Since there are \( n \) variables, there are \( 2^n \) ways of assigning these values to the \( n \) variables.

\( f(x_1, x_2, ... , X_n) \) there are \( 2^y \) (when \( y = 2^n \)) different switching functions of \( n \) variables.

For \( n = 1 \), the four functions of the variable \( A \) are

\[
\begin{align*}
&f_0 = 0, \quad f_2 = A \\
&f_1 = \bar{A}, \quad f_3 = 1
\end{align*}
\]

- 16 functions of the two variables A and B

\[
f(A, B) = \begin{cases} 
 1, & A = 1, B = 1 \\
 0, & A = 0, B = 0 \\
 1, & A = 0, B = 1 \\
 1, & A = 1, B = 0
\end{cases}
\]

- Dual functions

\[
\begin{align*}
f(A, B) &= A + \bar{B} \\
f(A, B) &= \bar{A} + B \\
f(A, B) &= \bar{A} B + \bar{B} \bar{A} \\
f(A, B) &= A B + \bar{A} \bar{B} \\
f(A, B) &= A B + \bar{A} \bar{B} \\
f(A, B) &= A \bar{B} + \bar{A} B \\
f(A, B) &= A B + \bar{A} \bar{B} \\
f(A, B) &= A \bar{B} + \bar{A} B \\
f(A, B) &= A B + \bar{A} \bar{B} \\
f(A, B) &= A \bar{B} + \bar{A} B \\
f(A, B) &= A B + \bar{A} \bar{B} \\
f(A, B) &= A \bar{B} + \bar{A} B \\
f(A, B) &= A B + \bar{A} \bar{B} \\
f(A, B) &= A \bar{B} + \bar{A} B \\
f(A, B) &= A B + \bar{A} \bar{B} \\
f(A, B) &= A \bar{B} + \bar{A} B
\end{align*}
\]
Switching function

### Truth Tables

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<th>(b)</th>
<th>(f(a, b) = a \cdot \overline{b})</th>
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### Algebraic Forms of Switching Function

- **SOP Forms**: Sum of products
  
  \[ f(A, B, C, D) = A \overline{B} C + \overline{B} \overline{D} + \overline{A} C D \]

- **POS Forms**: Product of Sums
  
  \[ f(A, B, C, D) = (A + B + C)(\overline{B} + C + \overline{D}) \]
Switching function

Algebraic Forms of switching function
- SOP Forms: Sum of products
  - SOP are constructed by summing (ORing) product (ANDed) terms.
  - Each product term is formed by ANDing a number of complemented and un-complemented variables.
  - Each variables called literal.

\[ f(A, B, C, D) = A\overline{B}C + \overline{B}D + \overline{A}C\overline{D} \]

Switching function

POS Forms: Product of Sums
- POS are constructed by taking the product(ANDing) of sum (OREd) terms.
- Each product term is formed by ORing a number of literal.

\[ f(A, B, C, D) = (\overline{A} + B + C)(\overline{B} + C + D) \]

Switching function - Minterm

Canonical Forms:
- Minterm: The product term is call minterm.
  - If the function is represented as a sum of mintermally, the function called canonical sum of products (canonical SOP) form.

\[ f_o(A, B, C) = \overline{A}B\overline{C} + AB\overline{C} + \overline{A}BC + ABC \]

- Each minterm is represented by an n-bit binarycode

Switching function - Minterm

- Each bit represent one of the variables of minterm as follows: Uncomplemented variable: 1
  Complemented variable: 0

\[ f_o(A, B, C) = \overline{A}B\overline{C} + AB\overline{C} + \overline{A}BC + ABC \]

<table>
<thead>
<tr>
<th>Minterm</th>
<th>Minterm Code</th>
<th>Minterm Number</th>
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<tbody>
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<td>\overline{A}B\overline{C}</td>
<td>010</td>
<td>\text{m}_3</td>
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<tr>
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<td>\text{m}_6</td>
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<tr>
<td>\overline{A}BC</td>
<td>111</td>
<td>\text{m}_7</td>
</tr>
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</table>

\[ f_o(A, B, C) = \text{m}_2 + \text{m}_3 + \text{m}_6 + \text{m}_7 \]

\[ f_o(A, B, C) = \sum_{\text{m}(2, 3, 6, 7)} m \]

The order of the variables in the functional notation is very important, since it determines the order of the bits of the minterm numbers.

\[ f_p(B, C, A) = \sum m(3, 3, 6, 7) \]
\[ f_p(A, B, C) = f_p(B, C, A) \]
\[ = m_2 + m_3 + m_6 + m_7 \]
\[ = \overline{A}BC + \overline{A}BC + \overline{A}BC + ABC \]
\[ = \overline{B}CA + \overline{B}CA + BCA + BCA \]
\[ = \overline{A}BC + ABC + \overline{A}BC + ABC \]
\[ = \sum m(1, 3, 5, 7) \]

<table>
<thead>
<tr>
<th>Row No.</th>
<th>Inputs</th>
<th>Outputs</th>
<th>Complement</th>
<th>( f_p(A, B, C) )</th>
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</tr>
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<td>( \leftarrow m_5 )</td>
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<td>( \leftarrow m_6 )</td>
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<td>( \leftarrow m_7 )</td>
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<td>7</td>
<td>1 1 1</td>
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<td>( \leftarrow m_7 )</td>
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Given the function

\[ f(A, B, Q, Z) = \overline{A}BQZ + \overline{A}BQZ + \overline{A}BQZ + \overline{A}BQZ \]

let us express the functions \( f(A, B, Q, Z) \) and \( f(A, B, Q, Z) \) in minterm list form.

\[ f(A, B, Q, Z) = \overline{A}BQZ + \overline{A}BQZ + \overline{A}BQZ + \overline{A}BQZ \]
\[ = m_0 + m_1 + m_6 + m_7 \]
\[ = \sum m(0, 1, 6, 7) \]

\( f(A, B, Q, Z) \) will contain the remaining 12 \((2^4 - 4)\) minterms. The minterm list for this function is

\[ f(A, B, Q, Z) = \sum m(2, 3, 4, 5, 8, 9, 10, 11, 12, 13, 14, 15) \]
Maxterm: The sum term is called maxterm.

- If the function is represented as a product of minterm only, the function is called canonical product of sum (canonical POS) form.

\[ f_y(A, B, C) = (A + B + C)(A + B + \bar{C})(\bar{A} + B + C) \]

Given the function \( f(A, B, C) = (A + B + \bar{C})(A + B + C)(\bar{A} + B + C)(\bar{A} + \bar{B} + \bar{C}) \), let us construct the truth table and express the function in both maxterm and minterm form.

\[ f(A, B, C) = \prod M(0, 1, 4, 5) = \prod M(1, 3, 5, 7) \]

<table>
<thead>
<tr>
<th>Row No. ((i))</th>
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</tbody>
</table>

\[ f_x(A, B, C) = \sum M(2, 3, 6, 7) = f_y(A, B, C) \]
Switching function - Maxterm

From the truth table for \( f(A, B, C) \), we observe that
\[
f(A, B, C) = \sum m(0, 2, 4, 6)
\]

Therefore,
\[
f(A, B, C) = \sum_{m(1, 3, 5, 7)} m_1 + m_2 + m_3 + m_7
\]
\[
= \bar{A}BC + A\bar{B}C + A\overline{B}C + ABC
\]

Consequently,
\[
f(A, B, C) = \bar{A}BC + A\bar{B}C + A\overline{B}C + ABC
\]
\[
= \bar{A}RC + \bar{A}RC + \bar{A}RC + \bar{A}RC
\]
\[
= (A + \overline{B} + C)(A + \overline{B} + C)(A + \overline{B} + C)(A + \overline{B} + C)
\]
\[
= M_1 M_2 M_3 M_4
\]

Therefore, we have algebraically shown that
\[
f(A, B, C) = \prod_{m(1, 3, 5, 7)} M(0, 2, 4, 6) = \sum m(0, 2, 4, 6)
\]

Derivation of Canonical Forms

Convert the following function to canonical SOP form:
\[
f(A, B, C) = AB + AC + \bar{A}C
\]

Let us apply Theorem 6a to each of the three product terms of this
\[
AB = AB\bar{C} + ABC = m_6 + m_7
\]
\[
AC = AC\bar{B} + ACB = A\bar{B}C + ABC = m_4 + m_6
\]
\[
\bar{A}C = \bar{A}C\bar{B} + \bar{A}CB = \bar{A}\bar{B}C + ABC = m_1 + m_3
\]

Therefore,
\[
f(A, B, C) = AB + AC + \bar{A}C
\]
\[
= (m_6 + m_7) + (m_4 + m_6) + (m_1 + m_3)
\]
\[
= \sum_{m(1, 3, 4, 6, 7)} m(0, 2, 4, 6)
\]

Switching function

Expand the following function to canonical POS form:
\[
f(A, B, C) = A(A + \bar{C})
\]

Theorem 6b can be applied as follows to produce maxterms.
\[
A - (A + \bar{B})(A + B) = (A + \bar{B} + C)(A + \bar{B} + C)(A + B -)
\]
\[
= M_1 M_2 M_3
\]
\[
A + \bar{C} = (A + \bar{C} + \bar{B})(A + \bar{C} + \bar{B}) = M_1 M_4
\]

Therefore,
\[
A(A + C) = (M_1 M_2 M_3)(M_1 M_4) = \prod_{m(0, 1, 2, 3)} M(0, 2, 4, 6)
\]
Switching function

- Incompletely Specified Function
  - The don’t care minterms will be labeled di instead of mi
  - The don’t care maxterms will be labeled Di instead of Mi

Switching function

Suppose that we are given a function $f(A, B, C)$ that has minterms $m_0, m_1, m_2$, and don’t-care conditions $d_4$ and $d_5$. We wish to express the function and its complement in both minterm and maxterm form and then reduce the function to its simplest form.

The minterm list form for this function is

$$f(A, B, C) = \sum m(0, 3, 7) + d(4, 5)$$

and the maxterm list is

$$f(A, B, C) = \prod M(1, 2, 6) \cdot D(4, 5)$$

$$\hat{f}(A, B, C) = \sum m(1, 2, 6) + d(4, 5)$$

$$\hat{f}(A, B, C) = \prod M(0, 3, 7) \cdot D(4, 5)$$

Switching function

$$f(A, B, C) = \bar{A}\bar{B}\bar{C} + \bar{A}BC + A\overline{B}\overline{C} + d(A\overline{B}\bar{C} + A\bar{B}C)$$

$$f(A, B, C) = \bar{A}\bar{B}\bar{C} + BC + d(A\overline{B}\bar{C} + A\bar{B}C)$$

$$f(A, B, C) = \bar{A}\bar{B}\bar{C} + BC + A\overline{B}\overline{C}$$

$$= \bar{B}\bar{C} + BC$$